MATH 130A Review: Factorials and Counting

Facts to Know

- Factorial
  \[ n! \]

  - **Recursive Defn**
    \[ 0! = 1 \quad \text{and} \quad n! = n \cdot (n-1)! \]
    \[ 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \]

  - **Direct Defn**
    \[ n! = (n)(n-1)(n-2)\ldots 1 \]

- \( k \)-Permutations of \( n \)-many distinct objects

  \[ 1, 2, 3, \ldots n \]

  \[ n=4, \ k=2 \]

  \[ (3, 2), \ (1, 4) \neq (4, 2) \]

- \( k \)-Combinations of \( n \)-many distinct objects

  \[ 1, 2, 3, \ldots, n \]

  \[ n=4, \ k=2 \]

  \[ \{3, 2\}, \ \{4, 1\} = \{3, 2\} \]

- The number of \( k \)-permutations of \( n \)-many distinct objects is

  \[ \frac{n!}{(n-k)!} \]

- The number of \( k \)-combinations of \( n \)-many distinct objects is

  \[ \frac{n!}{(n-k)! \cdot k!} = \binom{n}{k} \]
Examples

1. Let $k$ and $n$ be arbitrary natural numbers satisfying $0 < k < n$. Show that

- $\binom{0}{0} = \binom{n}{0} = \binom{n}{n} = 1$

\[
\binom{0}{0} = \frac{0!}{(0-0)! \cdot 0!} = \frac{1}{1} = 1
\]

\[
\binom{n}{0} = \frac{n!}{(n-0)! \cdot 0!} = 1
\]

\[
\binom{n}{n} = \frac{n!}{(n-n)! \cdot n!} = 1
\]

- $\binom{n}{k} = \frac{n}{n-k} \binom{n-1}{k}$

\[
\text{RHS} = \frac{n}{n-k} \cdot \frac{(n-1)!}{(n-1-k)! \cdot k!}
\]

\[
= \frac{n}{n-k} \cdot \frac{(n-1)!}{(n-k-1)! \cdot k!}
\]

\[
= \binom{n}{k} = \text{LHS}
\]

- $\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$
2. You have five digits (four fingers and one thumb) on one hand. You decide to pair two digits together. How many different combinations are there?

\[ \binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{5!}{(5-2)!2!} = \frac{5!}{3!2!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = 10 \]

3. Count how many rearrangements of the letters of your first name there are. This is an easy task for those with first names that do not repeat letters.

\[ \frac{n!}{(n-k)!} = \frac{7!}{(7-7)!} = 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \]